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Initial solidification in liquid metal film flow over a moving boundary

H.B. Löfgren*, H.O. Åkerstedt

Division of Fluid Mechanics, Department of Mechanical Engineering, Luleå University of Technology, S-971 87 Luleå, Sweden

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Abstract

The initial solidification problem of a two-dimensional liquid metal film flow over a heat extracting moving boundary is studied. Analytical solutions in the limit of large Peclet numbers are found. It is shown that the point of initial solidification depends on the Peclet number, the Biot number and the superheat. The initial growth of the solidified phase is found to have a quadratic dependence of the distance from the point of initial solidification. The results are applicable to continuous strip casters.

Keywords: Analytical; Liquid metals; Solidification

1. Introduction

The knowledge of solidification rates is of great importance in the field of casting technology. Many attempts have been made to derive exact and approximate mathematical models to simulate the solidification processes of metals. The far most studied case to date is that of unidirectional solidification. Exact analytic solutions are available for the limiting cases of engineering interest where heat flow is one-dimensional, mold–metal interface resistance is negligible, and the mold is either held at constant temperature or is very thick [1]. Approximate analytical solutions incorporating the effect of gaseous gaps generated at the interface between the solidified metal and the mold is presented by Davey [2].

The analytical treatment of the directional solidification problem appearing in connection with horizontal-belt-strip casting (HBSC) and planar-flow casters (PFC) have only recently received attention, see Löfgren and Åkerstedt [3], Carpenter and Steen [4] and Carpenter [5]. The liquid metal is here directed onto a conveyor belt or a spinning chill-wheel by a nozzle. A thin (1–15 mm, HBSC) (0.1–1 mm, PFC) solidified strip of aluminium or steel is then produced. Steen and Karcher [6] have recently reviewed the PFC technique. These castings are characterised by an approximately parallel film flow over a heat extracting moving boundary. At some distance downstream, the superheated melt reaches fusion temperature and a time independent solidification front emerges. Except from a small region close to the feeding point, the heat and fluid flow are only weakly coupled and interacts only through the shape of the solidification front.

This paper considers the initial solidification problem of continuous strip casters of the type where the liquid metal flow is two-dimensional and approximately parallel to the moving boundary. In Section 2, we analyse the initial heat problem of a superheated

* Corresponding author. Tel.: +46-920-72312; fax: +46-920-91047.

E-mail address: lofgren@mt.luth.se (H.B. Löfgren).

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Nomenclature

a	thickness of the liquid metal film flow (m)	$\bar{x}_s$	dimensionless distance to the point of initial solidification
$Bi_l = \frac{ha}{\kappa_l}$, $Bi_s = \frac{ha}{\kappa_s}$	the liquid and solid Biot numbers	y^*	vertical co-ordinate (m)
c	specific heat of solid metal (J/m^3 K)	$\bar{y} = \frac{y^*}{a}$, $y = Bi_l \bar{y}$, $Y = Bi_s \bar{y}$	dimensionless vertical co-ordinates
h	Newtonian heat transfer coefficient (W/m^3 K)	$\bar{z} = \bar{x} - \bar{x}_s$, $z = \frac{Bi_l^2}{Pe_l}\bar{z}$, $Z = \frac{Bi_s^2}{Pe_s}$	dimensionless translated horizontal co-ordinates
$Pe_l = \frac{Ua}{\alpha_l}$, $Pe_s = \frac{Va}{\alpha_s}$	the liquid and solid Peclet numbers	*Greek symbols*	
$Re = Ua/\nu$	Reynolds number	α_l, α_s	thermal diffusivity of liquid and solid metal (m^2/s)
$St = c(T_f - T_0)/\Delta h_f$	Stefan number		
$\bar{s} = \frac{s^*}{a}$, $s = Bi_l \bar{s}$, $S = Bi_s \bar{s}$	dimensionless thickness of the solidified phase	$\Delta = \frac{\Delta T}{T_f - T_0}$	dimensionless superheat
T	temperature field (K)	Δh_f	latent heat of fusion (J/m^3)
T_0	some reference temperature of the heat sink (K)	ΔT	superheat temperature above fusion temperature (K)
T_f	fusion temperature (K)	Γ	Gibbs–Thomson coefficient (m K)
U	main film velocity (m/s)	$\eta = y - s(z)$	dimensionless translated vertical co-ordinate
V	the velocity of the moving boundary (m/s)	κ_l, κ_s	thermal conductivity of liquid and solid metal (W/m K)
x^*	horizontal co-ordinate (m)		
$\bar{x} = \frac{x^*}{a}$, $x = \frac{Bi_l^2}{Pe_l}\bar{x}$	dimensionless horizontal co-ordinates	$\theta = \frac{T - T_0}{T_f - T_0}$	dimensionless temperature field

liquid metal film flow with a free surface and calculate the point of initial solidification. In Section 3, the initial solidification problem is considered. Conclusive remarks are give in Section 4.

2. The initial heat flow problem

Consider a two-dimensional continuous strip caster. Let the melt be a superheated pure metal with a free surface. In the region adjacent to the feeding point, both the temperature and the film flow is homogenous apart from thin boundary layers close to the moving boundary. In order to approximate the real problem, we may assume a slightly modified problem by neglecting the initially thin boundary layers generated within the feeding region, see Fig. 1. Furthermore, assuming the thermal boundary layer to be small with respect to film thickness, at the point of initial solidification, makes it possible to treat the liquid layer as semi-infinite in a first-order approximation.

It is convenient to introduce dimensionless variables by referring all lengths and velocities to the film thickness (a) and the main horizontal velocity (U). The dimensionless temperature is defined as

$$\theta_l = \frac{T_l - T_0}{T_f - T_0}, \quad (1)$$

where T_f is the fusion temperature and T_0 is some reference temperature of the heat sink. Then, in Cartesian co-ordinates, the dimensionless heat equation is given by

$$u\frac{\partial \theta_l}{\partial \bar{x}} + v\frac{\partial \theta_l}{\partial \bar{y}} = \frac{1}{Pe_l}\left\{\frac{\partial^2 \theta_l}{\partial \bar{x}^2} + \frac{\partial^2 \theta_l}{\partial \bar{y}^2}\right\}, \quad (2)$$

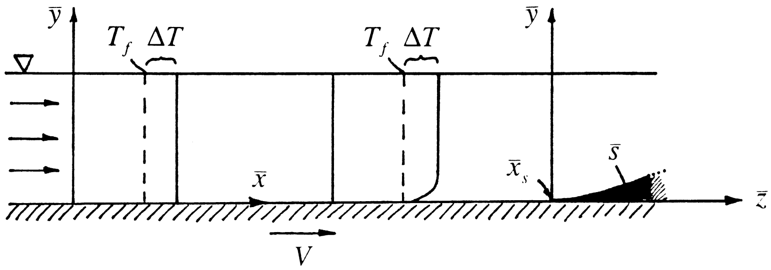

Fig. 1. Liquid metal film flow over a heat extracting moving boundary.

where $Pe_l = aU/\alpha_l$ is the liquid Peclet number and u and v are the horizontal and vertical velocities. See Table 1 for a closer view of the governing equations of this paper.

The approximate boundary conditions are

$$\theta_l(0, \bar{y}) = 1 + \Delta, \tag{3}$$

$$\theta_l(\bar{x}, \bar{y} \to \infty) \to 1 + \Delta, \tag{4}$$

$$\left.\frac{\partial \theta_l}{\partial \bar{y}}\right|_{\bar{y}=0} = Bi_l \theta_l(\bar{x}, 0), \tag{5}$$

where $\Delta \ll 1$ is the dimensionless superheat and $Bi_l = ha/\kappa_l$ is the liquid Biot number, assuming a constant Newtonian heat transfer coefficient h at the heat extracting boundary.

The condition at the point of initial solidification is

$$\theta_l(\bar{x}_s, 0) = 1, \tag{6}$$

neglecting curvature effects of the solidifying interface, which is valid whenever $d^2\bar{s}/d\bar{x}^2 \ll a(T_f - T_0)/\Gamma$. This assumption is validated in Section 3.2.

Searching for solutions in the limit of large Peclet numbers, we need to express the mathematical problem with derivatives of $O(1)$. An obvious scaling of the vertical co-ordinate is $y = Bi_l \bar{y}$ found by observing Eq. (5). Let a new horizontal co-ordinate be $x = \bar{x}/\delta(Bi_l, Pe_l)$. Inserting these new co-ordinates into Eq. (2) yields

$$\lim_{Pe_l \to \infty} \left\{ \frac{Pe_l}{Bi_l^2} \frac{u}{\delta} \frac{\partial \theta_l}{\partial x} + \frac{Pe_l}{Bi_l} v \frac{\partial \theta_l}{\partial y} - \frac{1}{Bi_l^2 \delta^2} \frac{\partial^2 \theta_l}{\partial x^2} \right\} = \frac{\partial^2 \theta_l}{\partial y^2}. \tag{7}$$

Knowing that $u = O(1)$ and $v = O(Re^{-1/2})$ from the theory of boundary layer flows [7], we see that the second term of the l.h.s. is negligible whenever $Bi_l/Pe_l \gg Re^{-1/2}$. This means that the boundary layer thickness of the velocity field is very much smaller than that of the temperature field. The velocity field can thereby, in this limit, be treated as homogenous. Without loss of generality, we may choose $\delta = Pe_l/Bi_l^2$ as $Pe_l \to \infty$, yielding the appropriate horizontal co-ordinate $x = Bi_l^2 \bar{x}/Pe_l$. The third term of the l.h.s is therefore of the order $O(Bi_l^2/Pe_l^2)$ and negligible in a first approximation whenever $Bi_l^2/Pe_l^2 \ll 1$. Hence, the sought solution is therefore restricted by $Bi_l/Pe_l \gg Re^{-1/2}$ and $(Bi_l/Pe_l)^2 \ll 1$.

This is a fair assumption for today's strip casters having Biot numbers ranging from 1 to about 1000, with typical Reynolds numbers of $\sim 10^4$ and Peclet numbers of ~ 100 for liquid aluminium and ~ 1000 for liquid steel.

The lowest order approximation with the new co-ordinates is then given by

Table 1
Governing equations and boundary conditions

The heat equation

$$u^* \frac{\partial T_l}{\partial x^*} + v^* \frac{\partial T_l}{\partial y^*} = \alpha_l \left\{ \frac{\partial^2 T_l}{\partial x^{*2}} + \frac{\partial^2 T_l}{\partial y^{*2}} \right\} \text{(liquid phase)}$$

$$V \frac{\partial T_s}{\partial x^*} = \alpha_s \left\{ \frac{\partial^2 T_s}{\partial x^{*2}} + \frac{\partial^2 T_s}{\partial y^{*2}} \right\} \text{(solid phase)}$$

Newton cooling at the moving boundary

$$\kappa_l \left.\frac{\partial T_l}{\partial y^*}\right|_{y^*=0} = h(T_l(x^*, 0) - T_0) \text{(liquid)}$$

$$\kappa_s \left.\frac{\partial T_s}{\partial y^*}\right|_{y^*=0} = h(T_s(x^*, 0) - T_0) \text{(solid)}$$

The temperature at the solidifying interface

$$T_l(x^*, s^*) = T_s(x^*, s^*) = T_f - R^{-1}\Gamma$$

Heat balance at the solidifying interface

$$\hat{\mathbf{n}} \cdot (\kappa_s \nabla^* T_s - \kappa_l \nabla^* T_l) = \Delta h_f V \frac{ds^*}{dx^*}$$

$$\frac{\partial\theta_l}{\partial x} = \frac{\partial^2\theta_l}{\partial y^2} \quad \text{for } x, y \geq 0, \tag{8}$$

with boundary conditions

$$\theta_l(0, y) = 1 + \Delta, \tag{9}$$

$$\left.\frac{\partial\theta_l}{\partial y}\right|_{y=0} = \theta_l(x, 0), \tag{10}$$

$$\theta_l(x, y\to\infty)\to 1 + \Delta, \tag{11}$$

and the condition at the point of initial solidification

$$\theta_l(x_s, 0) = 1. \tag{12}$$

The solution of the temperature field is found, using the Laplace transform in the x-direction, to be

$$\theta_l(x, y) = (1+\Delta)\left[\operatorname{erf}\left(\frac{y}{2\sqrt{x}}\right) + e^{x+y}\operatorname{erfc}\left(\sqrt{x} + \frac{y}{2\sqrt{x}}\right)\right], \tag{13}$$

which is self consistent for $x \gg \frac{1}{2}(\frac{Bi_l}{Pe_l})^2$, found by insertion into Eq. (7).

The distance to the point of initial solidification is now found using Eq. (12), giving

$$x_s = \frac{\pi}{4}\Delta^2 \quad \text{or} \quad \bar{x}_s = \frac{\pi}{4}\frac{Pe_l}{Bi_l^2}\Delta^2 \quad \text{valid for} \quad \sqrt{\frac{2}{\pi}\frac{Bi_l}{Pe_l}} \ll \Delta \ll 1. \tag{14}$$

3. The initial solidification problem

As the liquid metal starts to solidify, the heat flow becomes a coupled problem. The latent heat generated at the solidifying interface is extracted through a solid state, with thermal properties different from that of the liquid state. In order to determine the initial growth of the solid state, we study the heat problem in each phase separately and use the heat flux balance at the solidifying interface Eq. (15) as a solvability condition.

Let $\bar{s}$ be the dimensionless thickness, in units of a, of the solidified phase and assume weakly inclined solidification fronts. The dimensionless heat balance is then written as

$$\frac{1}{Bi_s}\left.\frac{\partial\theta_s}{\partial\bar{y}}\right|_{\bar{y}=\bar{s}} - \frac{1}{Bi_l}\left.\frac{\partial\theta_l}{\partial\bar{y}}\right|_{\bar{y}=\bar{s}} = \frac{Pe_s}{StBi_s}\frac{d\bar{s}}{d\bar{z}}, \quad \bar{z} = \bar{x} - \bar{x}_s. \tag{15}$$

This assumption is validated by the results in Section 3.2.

3.1. The liquid metal heat flux at the solidifying interface

In this section, we seek the distribution of the liquid metal heat flux at the solidifying interface close to the point of initial solidification. The method of solution is to transform the problem so that the power of Fourier analysis is applicable.

Introduce the new co-ordinates

$$z = x - x_s \quad \text{and} \quad \eta = y - s(z), \tag{16}$$

where $s = Bi_l\bar{s}$. Assume that $ds/dz \ll 1$ in the limit of small z and large Peclet numbers. This is validated later in Section 3.2. It is easily seen that the form of the heat equation (8) remains with the new set of co-ordinates. Then, by introducing the function

$$\Theta = \theta - 1 \quad \text{for } \eta \geq 0, \tag{17}$$

and assuming that $\Theta(z, \eta) = -\Theta(z, -\eta)$ in order to ensure the boundary condition $\Theta = 0\,(\theta = 1)$ at $\eta = 0$, we have generated the problem

$$\frac{\partial\Theta}{\partial z} = \frac{\partial^2\Theta}{\partial\eta^2}; \quad -\infty < \eta < \infty, z \geq 0, \tag{18}$$

with the initial temperature distribution

$$\Theta(0, \eta) = \frac{\eta}{|\eta|}\left\{(1+\Delta)\left[\operatorname{erf}\left(\frac{|\eta|}{\Delta\sqrt{\pi}}\right) + e^{\frac{\pi}{4}\Delta^2+|\eta|}\operatorname{erfc}\left(\frac{\Delta\sqrt{\pi}}{2} + \frac{|\eta|}{\Delta\sqrt{\pi}}\right)\right] - 1\right\}, \tag{19}$$

found from Eqs. (13) and (14). The solution of Eqs. (18) and (19) is given by

$$\Theta(z, \eta) = \frac{1}{\sqrt{4\pi z}}\int_{-\infty}^{\infty}\Theta(0, \eta')e^{-\left[(\eta-\eta')^2\right]/4z}\,d\eta', \tag{20}$$

see [8]. Then, for weakly inclined solidification fronts, the heat flux is

$$\left.\frac{\partial\theta_l}{\partial y}\right|_{y=s(z)} = \left.\frac{\partial\Theta}{\partial\eta}\right|_{\eta=0} = \frac{1}{4\sqrt{\pi}z^{3/2}}\int_{-\infty}^{\infty}\eta'\Theta(0, \eta')e^{-\eta'^2/4z}\,d\eta' \overset{\text{symmetry}}{=} \frac{1}{2\sqrt{\pi}z^{3/2}}\int_{0}^{\infty}\eta'\Theta(0, \eta')e^{-\eta'^2/4z}\,d\eta'. \tag{21}$$

Close to the point of initial solidification, $z\to 0_+$, the exponential term of the integrand rapidly goes to zero even for small η'. This makes it possible to approximate the initial distribution (19) by its Taylor expansion, given by

$$\Theta(0,\eta) = \eta + \left(1 - \frac{2}{\pi}\frac{1+\Delta}{\Delta}\right)\left(\frac{\eta^2}{2} + \frac{\eta^3}{6}\right) + O(\eta^4); \quad \eta \geq 0. \tag{22}$$

Inserting Eq. (22) into Eq. (21) gives the heat flux distribution

$$\frac{1}{Bi_l}\frac{\partial\theta_l}{\partial\bar{y}}\bigg|_{\bar{y}=\bar{s}(\bar{z})} = \frac{\partial\theta_l}{\partial y}\bigg|_{y=s(z)} = 1 + \left(1 - \frac{2}{\pi}\frac{1+\Delta}{\Delta}\right)z + O(z^2) \approx 1 - \frac{2}{\pi\Delta}z = 1 - \frac{2}{\pi\Delta}\frac{Bi_l^2}{Pe_l}\bar{z}. \tag{23}$$

Now, knowing the heat flux from the liquid makes us ready to investigate the heat flow problem in the solid phase and to see if our assumption of a weakly inclined initial solidification front is appropriate.

3.2. The heat problem in the initially thin solid phase

In this section, we solve the heat flow problem for the initially thin solidified phase, assuming a weakly inclined solidification front. The dimensionless heat flow problem for the solid phase is therefore formulated as

$$\frac{\partial\theta_s}{\partial\bar{z}} = \frac{1}{Pe_s}\left\{\frac{\partial^2\theta_s}{\partial\bar{z}^2} + \frac{\partial^2\theta_s}{\partial\bar{y}^2}\right\}, \tag{24}$$

with the boundary conditions

$$\frac{\partial\theta_s}{\partial\bar{y}}\bigg|_{\bar{y}=0} = Bi_s\theta_s(\bar{z},0), \tag{25}$$

$$\theta_s(\bar{z},\bar{s}) = 1, \tag{26}$$

and the heat balance at the solidifying interface

$$\frac{1}{Bi_s}\frac{\partial\theta_s}{\partial\bar{y}}\bigg|_{\bar{y}=\bar{s}(\bar{z})} - 1 + \frac{2}{\pi}\frac{Bi_l^2}{Pe_l}\frac{\bar{z}}{\Delta} + \cdots = \frac{Pe_s}{StBi_s}\frac{d\bar{s}}{d\bar{z}}, \tag{27}$$

from Eqs. (15) and (23).

Then again, when seeking solutions in the limit of large Peclet numbers, we need to express the mathematical problem with derivatives of $O(1)$. In the same way, as in Section 2, we may introduce new variables as $Z = Bi_s^2\bar{z}/Pe_s$, $Y = Bi_s\bar{y}$ and $S = Bi_s\bar{s}$. The asymptotic problem in the thin solid shell is then given by

$$\frac{\partial\theta_s}{\partial Z} = \frac{\partial^2\theta_s}{\partial Y^2}, \quad Z \geq 0,\ 0 \leq Y \leq S(Z) \tag{28}$$

$$\frac{\partial\theta_s}{\partial Y}\bigg|_{Y=0} = Bi_s\theta_s(Z,0), \tag{29}$$

$$\theta_s(Z,S) = 1, \tag{30}$$

$$\frac{\partial\theta_s}{\partial Y}\bigg|_{Y=S(Z)} - 1 + \frac{2}{\pi}\frac{Bi_l^2}{Bi_s^2}\frac{Pe_s}{Pe_l}\frac{Z}{\Delta} + \cdots = \frac{1}{St}\frac{dS}{dZ}, \quad \text{as } Z \to 0_+ \tag{31}$$

where the ratios Pe_l/Pe_s and Bi_l/Bi_s are of $O(1)$.

In order to determine the temperature field in the region close to the point of initial solidification, we assume a Taylor expansion of θ_s for small Z and Y, i.e.

$$\theta_s(Z,Y) = \theta_0 + A_1Z + A_2Y + B_1Z^2 + B_2ZY + B_3Y^2 + C_1Z^3 + C_2Z^2Y + C_3ZY^2 + C_4Y^3 + \text{H.O.T.} \tag{32}$$

By inserting Eq. (32) into Eqs. (28)–(30), a straightforward identification gives

$$A_1 = B_2 = 2B_3 = 6C_4 = \cdots$$

$$A_2 = \theta_0 = 1 = \cdots$$

$$B_1 = C_2 = C_3 = \cdots$$

$$A_1Z + S + B_1Z^2 + B_2ZS + B_3S^2 + \cdots = 0 \tag{33}$$

To commence the analysis, we know that in the case of no superheat in the melt, the initial growth of the solid phase is linear [3]. It is therefore reasonable to assume that $S = kZ^q + \cdots$ as $Z \to 0_+$, where k is some constant and $q \geq 1$. The only proper choice fulfilling both Eqs. (31) and (33) is $q = 2$, giving the solution

$$\theta_s = 1 + Y - \frac{St}{\pi\Delta}\frac{Bi_l^2}{Bi_s^2}\frac{Pe_s}{Pe_l}\left(Z^2 + Z^2Y + ZY^2\right) + \text{H.O.T} \tag{34}$$

and

$$S = \frac{St}{\pi\Delta}\frac{Bi_l^2}{Bi_s^2}\frac{Pe_s}{Pe_l}Z^2 + O(Z^3) \quad \text{as } Z \to 0_+. \tag{35}$$

This validates the assumption of weakly inclined solidification fronts in the vicinity of the point of initial solidification.

Let us now investigate if the curvature of the solidifying interface effects the fusion temperature. We know that this effect is negligible as long as $d^2\bar{s}/d\bar{x}^2 \ll$

$a(T_f - T_0)/\Gamma$. Using Eq. (35) yields an additional restriction for the superheat, namely

$$\Delta \gg \frac{2St}{\pi} \frac{Bi_l^2 Bi_s}{Pe_l Pe_s} \frac{\Gamma}{a(T_f - T_0)}, \quad (36)$$

where $\Gamma/[a(T_f - T_0)]$ is typically very much smaller than unity. This is thereby a far weaker restriction than that for consistency of Eqs. (13) and (14), which is

$$\frac{Bi_l}{Pe_l} \gg \frac{1}{\sqrt{Re}}; \quad \left(\frac{Bi_l}{Pe_l}\right)^2 \ll 1; \quad \sqrt{\frac{2}{\pi}} \frac{Bi_l}{Pe_l} \ll \Delta \ll 1. \quad (37)$$

Consequently, the curvature effect is negligible within the given parameter interval of this analysis.

4. Conclusions

In this paper, we analyse the two-dimensional initial solidification problem of a superheated pure liquid metal, appearing in connection with the continuous strip casting process. Analytical solutions are found in the limit of large Peclet numbers. It is shown that the distance to the point of initial solidification depends on the Peclet number, the Biot number and the superheat. Furthermore, the initial growth of the solid phase is found to have a quadratic dependence of the distance from the point of initial solidification. This is to be compared with the case of no superheat where the initial growth is linear [3].

The results of this paper are intended to give a first-order approximation to the initial solidification process that can be used as an independent check of numerical calculations. This would also be a good approximation for the case of low-alloyed metals for which the mushy two-phase zone can be neglected. An interesting extension to this work is to consider the effect of solute concentrations. This should decrease the fusion temperature and thereby postpone the point of initial solidification.

Acknowledgements

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